School of Graduate Studies
Bemidji State University
1500 Birchmont Dr NE, #27
Bemidji, MN 56601-2699
218-755-2027

COMMON MISCONCEPTIONS AND INSTRUCTIONAL BEST PRACTICES FOR TEACHING FRACTIONS TO STUDENTS IN GRADES 3, 4, AND 5

by

Dan Fairchild

A Research Paper Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE IN MATHEMATICS

BEMIDJI STATE UNIVERSITY
Bemidji, Minnesota, USA

May 2013

STATEMENT BY THE AUTHOR

Signed: ________________________

APPROVAL BY RESEARCH PAPER ADVISOR

THIS RESEARCH PAPER HAS BEEN APPROVED ON THE DATE SHOWN BELOW:

____	______________________
Dr. Glen Richgels, Committee Chair Professor, Department of Mathematics and Computer Science	Date
____	______________________
Dean, School of Graduate Studies	Date

COMMON MISCONCEPTIONS AND INSTRUCTIONAL BEST PRACTICES FOR TEACHING FRACTIONS TO STUDENTS IN GRADES 3, 4, AND 5

Dan Fairchild

This research paper will identify common misconceptions related to the teaching and learning fractions for students in grades three, four, and five. Different fraction subconstructs will be identified and instructional best practices will be identified for the teaching and learning of comparing fractions, ordering fractions, adding fractions, and subtracting fractions.

Approved by:

______________________________ ______________
Committee Chair Date

____
Committee Member

____
Committee Member

____
Graduate Faculty Representative

ACKNOWLEDGMENTS

I would like to thank the professors and staff at Bemidji State University for giving me guidance along the research process. Dr. Glen Richgels, as well as Dr. Todd Frauenholtz and Dr. Derek Webb provided support and inspired confidence in my abilities. A big thank you goes to my wife for keeping me motivated and organizing our lives during this hectic time!

TABLE OF CONTENTS

Chapter 1: Introduction

Going through school, I was the type of student who loved math. It came easy to me, it made sense, and I had fun; however, in my third, fourth, and fifth grade years, we visited the topic of fractions. This instantly changed my view on the subject of mathematics. Suddenly, math was no longer easy, it did not make sense, and I did not enjoy the subject. Just the word "fractions" made me cringe. All of the rules I learned had gone out the window. I now had to do things completely different, the rules had changed, and I did not know why. My confidence level dropped, and so did my grades.

I do not believe I was the only student who felt discouraged. In fact, I know I was not because I now have my own fifth grade classroom and I see this happen to several of my students each year. I have students who come in very confident in their mathematical skills and abilities only to reach the topic of fractions and become disheartened. They lose their confidence and their attitude about mathematics changes instantly. It changes a subject that once was fun but now is like to a trip to the dentist where you will be leaving with fewer teeth than you had when entering. In conversations with my former teachers and current colleagues, this is not a rare occurrence. In the Trends in International Mathematics and Science Study of 2007, United States fourth grade students ranked eleventh out of thirty-six possible nations in mathematics (Martin, Mullis & Foy, 2008). Many of the nations the U.S. students ranked higher than are considered poverty nations. This indicates a serious concern in the performance of our elementary students.

Our district's adopted textbook series teaches fractions in a very abstract way, with mathematical symbols, procedures, and "rules" to follow based on what we are trying to accomplish with fractions. We spend most of our time focusing on processes and very little of it teaching comprehension of what a fraction really is. We know young students go through various stages of development and very rarely, if ever, are elementary school students ready for this style of instruction. I now believe, because most students are not ready for the abstract representation of fractions, our students are set up for failure.

The importance of using concrete models with elementary students is supported by the work of Piaget and Bruner (Post, 1992). While taking graduate classes at Bemidji State University, a mathematics professor introduced me to rational number instructional materials that changed the way I view teaching and learning fractions. These materials teach the concept of fractions in the way students learn. They spend a great deal of time conceptualizing what fractions are and, only after students have grasped this concept, do students move on to other skills and stages of representation. Various types of manipulatives are introduced to students and multiple solution paths are encouraged. After the first year of teaching using these materials in my classroom, I saw attitudes improve, confidence rise, and assessment scores increase. This really lit a fire for me in the topic of rational number instruction.

Significance of the Research Problem

With current standardized testing in Minnesota, the fifth-grade version of the Minnesota Comprehensive Assessments (MCA's) has roughly forty percent of its questions related to the topics of fractions, decimals, and percents ("Mathematics test specifications," 2010). Knowing that the grade level at my current school has an average passing rate of 35.25% on the Minnesota Comprehensive Assessments (MCA's) over the last four years (www.greatschools.org), I thought, if we can find a way to make fractions instruction understandable, meaningful, and relatable to our students, then we can come out on the winning-side of this challenge more often.

Research Questions

1. How many different conceptions of fractions are there?
    - What does each fraction concept entail?
    - What foundations and skills do students need in order to show mastery of each concept?
2. What are some common misconceptions encountered during instruction?
3. What are instructional best practices for teaching students in third, fourth, and fifth grades fraction concepts?

Limitations and Assumptions

This study is limited to research on fractions instruction and misconceptions for students in grades three, four, and five. I am also limiting my research to topics of fraction instruction that relate to concepts that are expected to be taught to students in grades 3, 4, and 5 based on Minnesota State Academic Standards.

This study assumes all students have the capacity to learn fraction concepts at a high level, given appropriate instruction. I am also assuming there is a manipulative with which each student can become comfortable and successful. I am looking for fraction instruction pedagogy that will make fraction learning fun, interesting, and effective for students in a way that will not inhibit confidence, but will instead inspire it.

Definition of Terms

Denominator- The number q in a fraction p/q, i.e., the divisor.

Density- Between two rational numbers on a number line there is another rational number.

Dividend- A number to be divided.

Divisor- The number by which a dividend is divided.

Fraction Circles- Fraction circles are a set of nine circles of various colors. Each circle is broken into equal fractional parts and uses the same-sized whole.

Integer- Positive or negative counting numbers or zero.

Manipulative - Any concrete objects that allow students to explore an idea in an active, hands-on approach.

Numerator- The number p in a fraction p/q, i.e., the dividend.

Partitive Division- The number of groups the dividend will be split into is known, but the size of each group is unknown (Petit, Laird, & Marsden, 2010).

Quotitive Division- The divisor indicates the size of the group and the quotient is the number of equal sized groups (Roche & Clarke, 2009).

Rational number- A rational number is a number that can be expressed as a fraction p/q where p and q are integers and $q \neq 0$. A rational number p/q is said to have numerator p and denominator q.

Traditional Mathematics Curricula- Curricula developed which emphasized memorizing rules for computation and repetition with little importance put on understanding why concepts work.

Unit- A single undivided entity or whole.

Summary Statement

The comprehension of rational numbers is a critical component to the success of mathematics instruction. With an increased proportion of fraction questions on standardized assessments, teachers need to find methods to adapt our teaching to meet the learning needs of students. This research is aimed at helping identify weaknesses in current fraction instruction and identifying strategies that can be utilized to improve academic success for students in grades three, four, and five.

Chapter 2: Summary of Research Sampling

It has been recorded in both state and national assessments that students are learning very little, if anything, about rational numbers (Lamon, 2001). “More than 90% of students perform fractions-related tasks at a level far below anything that might remotely be considered a ‘world-class standard’”(Neimi, 1996, pg. 77).

The concept of rational numbers has many different aspects to consider. The following research discusses the types of fractions students will see, common misconceptions regarding fraction instruction, and instructional best practices for skills fifth grade students in Minnesota are expected to have mastered; specifically comparing and ordering fractions, adding fractions, and subtracting fractions.

How many types of fractions concepts are there?

Studies have suggested one of the main reasons fractions are difficult to understand is they are so expansive. Rational numbers can be categorized into five interrelated categories: part-whole, ratio, operator, quotient, and measure (Charalambous & Pitta-Pantazi, 2007). The following paragraphs will analyze each specific category in order to understand how to better help students master the concept.

Part-Whole

This focus of fractions is defined as a situation in which a quantity or a set of objects are divided into parts of equal size (Lamon, 1999; Marshall, 1993). In order for students to have mastered this category, they should understand the unit needs to be broken into equal sized parts and should be able to perform the partitioning. (Charalambous & Pitta-Pantazi, 2007). There are critical ideas associated with the relationship between parts and the whole such as: the parts, when taken together, must complete the unit, the more parts the unit is divided into, the smaller the parts become, and the relationship between the parts and the unit is constant, regardless of the size, shape, arrangement, or orientation of the equal-sized parts (Charalambous, & Pitta-Pantazi, 2007).

Ratio

The ratio category of fractions looks at the comparison between two amounts and is considered comparative, rather than a number (Behr & Post, 1992; Carraher, 1996). To fully grasp fractions as ratios, students need to build the idea of relative amounts (Lamon, 1993; Marshall, 1993). Students should also understand that when the two compared amounts change, they change together, and the relationship between the two remains the same (Charalambous & Pitta-Pantazi, 2007). Similarly, learners need to realize when the two amounts in a ratio are multiplied by the same nonzero number, the value of the ratio remains the same (Charalambous & Pitta-Pantazi, 2007). Fraction equivalence comes from this category and must be developed in order to have a full grasp of the concept (Marshall, 1993).

Operator

In this interpretation of rational numbers, fractions are looked at as functions applied to a number or set (Behr, Harel, Post, & Lesh, 1993; Charalambous, & Pitta-Pantazi, 2007; Marshall, 1993). In order to master the operator facet, students should be able to understand a fractional multiplier in a variety of ways (eg., 4/7 could be understood as either 4 x [1/7 of a unit] or 1/7 x [4 units]). Students should be able to relate inputs and outputs (for example, a 4/7 operator results in transforming an input capacity of 7 into an output capacity of 4) (Charalambous, & Pitta-Pantazi, 2007).

Quotient

Within the quotient category, any fraction can be seen as the result of a division circumstance. In particular, the fraction 4/5 describes the value attained when 4 is divided by 5 (Charalambous, & Pitta-Pantazi, 2007; Kieren, 1993). Activities designed to help students build this concept of fractions include problems of finding “fair shares” of objects, such as pizzas or pancakes (Behr & Post, 1992; Marshall, 1993; Streefland, 1993).

Unlike the part-whole category, the quotient personality reflects two different measure spaces, for example, three *pizzas* are shared among four *friends*. Also, since the attained result refers to a numerical value and not to the number of parts developed by the fair-sharing activity, the quotient facet has no limits regarding the size of the fraction. “The numerator can be less than, equal to, or greater than the denominator, and consequently, the amount that results from the fair-sharing activity can be less than, equal to, or greater than the unit” (Charalambous & Pitta-Pantazi, 2007, pg. 299).

Besides developing the fair sharing idea, students also need to be able to associate fractions with division and realize the role of the dividend and the divisor in the operation (Charalambous & Pitta-Pantazi, 2007; Lamon, 1999). Mastering the quotient construct requires learners to develop an understanding of partitive and quotitive division (Kieren, 1993; Marshall, 1993). Partitive division puts importance on the amount each participant gets, while quotitive division stresses the number of equal parts that can be made when separating an amount into equal parts (Charalambous & Pitta-Pantazi, 2007).

Measure

When we discuss the measure concept of fractions, we are looking at a distance. A unit fraction is defined (a fraction that has a numerator of one) and used repeatedly to determine a distance from a preset starting point (Lamon, 2001; Marshall, 1993). For example, 5/7 relates to the distance of five (1/7 –units) from a given point.

Lamon (1999) considers carrying out groups other than halving as a necessary skill for the development of the measure construct of fractions. She also suggests that a strong comprehension of the measure construct requires the understanding that between any two amounts there is an infinite number of fractions (Lamon, 1999).

Common Misconceptions Relating to Fractions

There have been several suggestions generated as to why students perform low on fractions, including:

- the complex nature of fraction concepts (Streefland, 1991),
- emphasis placed on teaching procedures rather than conceptual understanding (Moss & Oliver, 1999),
- teaching computation before understanding (Aksu, 1997),
- incorrect knowledge of rote procedures (Lamon, 2001),
- troubles identifying units (Baturo, 2004), and
- interference from whole-number knowledge (Cramer, Post, & del Mas, 2002).

Many reasons have been listed, but this paper will focus on the two groups that most of these can fall into: the complexity of fractions and rote teaching/learning practices.

Romberg and Carpenter (1986) suggest a need to rethink the content of the school mathematics program based on both the research on how students learn mathematics and research on teaching mathematics. But, before we get too far, we need to look at why fraction instruction has traditionally been performed so abstractly.

> Lamon (2001) states, “fractions and ratios were once the tools of clerks and bookkeepers. Ciphering was learned by meticulously copying the work of a master, and it had disciplinary as well as practical value. The computation of fractions assumed an important role in the elementary school mathematics curriculum with the expansion of business and commerce during the Industrial Revolution of the eighteenth and nineteenth centuries (National Council of Teachers of Mathematics [NCTM] 1970). In the early 1900s, psychology started influencing instruction, and part/whole relationships began to be used to introduce the vocabulary and symbolism of fractions. However, students were afforded only a brief encounter with partitioning activities to build meaning because a time-efficient route to the formal symbolic computation best served the needs of society at that time” (p.151).

Many school districts are still using variations of curricula created during the 1950’s when this style of education was current. Also, many of today’s teachers went through a school system that employed this style of education. The result is fractions are a topic that many teachers have difficulty understanding and teaching (Behr et al., 1993; Behr, Lesh, Post, & Silver, 1983).

Rote Instruction/Learning

Chan, Leu, & Chen state "[r]ote teaching and learning remains a major cause of low performance" (2007, pg. 28). While teaching fractions, a common error is to have students begin with operations before they have the necessary background knowledge to benefit from such computations (Aksu, 1997). In elementary schools using traditional mathematics curricula, typically more than 85% of the instructional time is used to teach and practice computational procedures and less than 15% is devoted to the development of the concept of fractions (Niemi, 1995).

The NCTM and many researchers have advised that conceptual understanding be developed before computational fluency (Bezuk & Cramer, 1989; Cramer, Post, & del Mas, 2002). Students must understand the meanings of fractions before performing operations with them. For a traditional mathematical algorithm for make sense to a learner, it must signify the organization of mental operations and conventional notes (Saenz-Ludlow, 1995). Lamon (2001) declares current fraction instruction could be improved simply by providing more time and opportunities to build understanding without providing algorithms for our students.

Complexity of Fractions

According to research by Behr, Wachsmuth, Post, & Lesh (1984), some students inappropriately use whole-number reasoning, seeing a fraction as two whole numbers instead of a single quantity. With whole numbers, the idea of counting the next number makes sense. The number after one is two. This idea does not work with fractions.

Because of this, counting rational numbers does not make sense, which makes fractions difficult to order (Behr & Post, 1988). One result of this is students tend to apply a familiar algorithm, correct or not, learned from whole-number arithmetic into fractions arithmetic (Niemi, 1995).

Sometimes teachers focus on a limited number of constructs while teaching fractions. There is a disproportionate amount of classroom time spent teaching children the part/whole subset of fractions. Lamon (2001) states, “[m]athematically and psychologically, the part/whole interpretation of fractions is not sufficient as a foundation for the system of rational numbers (pg. 158).” Teachers may be inadvertently reinforcing inappropriate whole number reasoning by only providing opportunities for students to solve part-whole relationship problems (Petit, Laird, & Marsden, 2010).

Instructional Best Practices

The National Council of Teachers of Mathematics (1989, pg. 34) has stated “learning to solve problems is the principle reason for studying mathematics.” One major step toward achieving this goal would be scaffolding students toward understanding comparisons and being able to conceptualize a single idea across a wide range of representations.

In support of this statement, Lesh et al. stated that mathematical understanding can be defined as the ability to translate among various representations of an idea (1987). “A large part of mathematics can be understood as the building of models, solving problems within models, and then translating these solutions back into the real world (Orton, et al., 1995, pg. 64).” The development of children’s understanding of rational number concepts has been related to three characteristics of children’s thinking (Orton et al., 1995). The first is a child's ability to translate between different expressions of rational number concepts. The second characteristic is that a child can translate among different ways to express a concept within the same form of representation. The third characteristic is the child's ability to progressively move away from concrete representations of rational numbers to more symbolic models (Orton et al., 1995).

In order to understand rational numbers, a student must have a strong foundation in the four operations with whole numbers and an understanding of measurement concepts (Behr & Post, 1992). Lamon (2001, pg. 149) states "[t]he ultimate goal of fractions instruction is to help students understand fractions as numbers in their own right, and, as such, as objects that can be manipulated with arithmetic."

This paper will now examine components of quality fraction instruction including: conceptualization, concrete models to use with students, problem-solving, and successful strategies used to compare fractions.

Conceptualization

Achieving this goal takes a significant amount of time. Lamon (2001) discovered it could take students up to two years to become comfortable and flexible in answering higher-order thinking questions related to fractions. Studies done by the team of researchers that created the Rational Number Project (Behr et al., 1984) showed that students, who received up to eighteen weeks of intensively-planned instruction which used multiple representations of fraction symbols, showed a "substantial lack of understanding" and could still only solve questions that did not deal with the application of fraction knowledge of a new situation.

In a study performed by Karplus, Pulos, and Stage (1983), student classes were introduced to different types of instruction of rational numbers. The goal of this study was to determine if there was a more successful order to teach fractions than the current text books were using. Students were placed into six groups. One of the six groups was taught using traditional part/whole instruction. The other five groups were taught never receiving rules or algorithms, rather being given challenges and being expected to engage in problem solving situations. Each of these five groups was introduced to a different facet of rational numbers and was taught to reason with fractions, never carrying out the computation of fractions. The percentage of proportional reasoners, as well as group achievement, in the five study groups was superior to the control group.

> "Using different interpretations of rational numbers, children acquired meanings and processes in different sequences, to different depths of understanding, and at different rates. The time-honored learning principle of transferability was robust. Children transferred their knowledge not only to unfamiliar circumstances, but also to other interpretations that they had not been directly taught (Lamon, 2001, pp. 159-60)."

Problem Solving

"Problem solving is not a distinct topic, but a process that should permeate the entire program and provide the context in which concepts and skills can be learned" (NCTM, 1989, pg. 23). If students are to truly believe that mathematics, and fractions, in particular, makes sense, then instruction must allow students to invent their own ways to operate on fractions (Huinker, 1998). "Current instruction in fractions grossly underestimates what children can do without our help. They have a tremendous capacity to create ingenious solutions when they are challenged" (Lamon, 2001, pg. 153). Moss & Case (1999) conclude that by using meaningful contexts, students develop "fraction language."

Models

In order to teach the conceptualization of rational numbers, we need to introduce our students to multiple models. Dienes' Perceptual Variability Principle (1960) suggests in order to allow as much opportunity as possible for individuality in concept developments, as well as to encourage children to gather the mathematical essence of an abstraction, the same conceptual idea should be presented in as many perceptual equivalents as possible (pg. 32). Each of the five subconstructs listed in this chapter (part-whole, ratio, operator, quotient, and measure) has "its own set of representations and operations, models that capture some-but not all- of the characteristics of the field of rational numbers" (Kieren, 1976). With the goal of providing as broad and deep a foundation as possible for meanings and operations with rational numbers, Lamon (2001) concludes not all the subconstructs of rational numbers are equally good starting points. Behr & Post (1992) recommend the concept of fractions could be established with the foundation of continuous models and then a transfer could be made to discrete models. Continuous models refer to area and measurement interpretations, while discrete models involve the use of counters to show the unit (Cramer & Wyberg, 2009).

Fraction Circles

The first collection of lessons teaching rational numbers created by the Rational Number Project (RNP) uses fraction circles to develop the concept of fractions. Fraction circles are circles that are sorted by color and cut into different-sized pieces. This model is designed for students to create mental images enabling them to learn fraction-size (Cramer & Henry, 2002). Students can then take their understanding of fraction-size to perform operations on fractions meaningfully (Cramer & Post, 2002).

To model addition and subtraction of fractions using this model, students first show both fractions on a unit circle. Then, students are challenged by finding a color that covers both fractional amounts. The need for a common denominator (common color) is more apparent with this model than with discrete models and the unit may be defined as the circle (Cramer & Wyberg, 2009). "The fraction circle model used in combination with RNP activities was the most powerful of the models. During interviews, students consistently referred to fraction circles as the model that helped them order fractions and estimate the reasonableness of fraction operations" (Cramer & Henry, 2002, pg. 42).

Number Lines

Although the number line is introduced to students in most elementary school textbooks, its potential for student learning has not been seized by educators or researchers (Saxe, Shaughnessey, Shannon, Langer-Osuna, Chinn, & Gearhart, 2007, pg.1). Teaching rational numbers in the measurement construct was found to be the most valuable starting point in a different study, with learners having the highest level of transferability among fractional categories (Lamon, 2001, pg. 160).

Researchers suggest number lines can help build an understanding of the magnitude of fractions, and can be used to build concepts of equivalence and the density of rational numbers (Behr & Post, 1992). Furthermore, using problem solving as an approach to teach fractions allows situations to be chosen in a manner where the number line model makes sense (Moss & Case, 1999).

The use of number lines can be an essential tool used in the elementary classroom; however, the introduction of this model needs to be a process. Before working solely with number lines, evidence shows that translating the linear feature of other part/whole models to number lines may help explain improvement in the future use of number lines (Bright, Behr, Post, & Wachsmuth, 1998). The use of number lines should be introduced after other part-whole models that students can actually manipulate, such as fraction circles, because it is a more abstract representation for fractions (Cramer & Wyberg, 2009).

When first introducing number lines to young students, research suggests that teachers show number lines with whole numbers only (Petit et al., 2010). The focus should be to have students think about their number lines with whole numbers proportionally, not sequentially. Once students have built the idea of evenly spaced parts on number lines, then rational numbers may be introduced.

As with all teaching models, Behr and Post (1992) suggest teachers become familiar with the full knowledge of difficulties students may encounter. Some difficulties include: students reverting to their whole number reasoning and placing fractions on the number line in order of its value in their numerator or denominator, having difficulty defining the unit, and becoming distracted by the symbols and tick marks which define the units and sub-units (Behr et al., 1983; Petit et al., 2010).

Paper Strips

Manipulating paper strips is an activity students can participate in to focus on partitioning (Behr & Post, 1992). For this activity, students take a piece of paper and work on folding the continuous area into halves, then fourths, eighths, etc. After each fold, Behr & Post (1992) recommend that students unfold the paper to see how many equal-sized parts they have made on the paper. Once students have become comfortable making these partitions we can look at finding equivalents. This can be done by looking at the shaded area of the unit and making more folds in the paper. Paper folding is an excellent resource for showing the multiplication of fractions, but, since it is not a 5th grade standard in Minnesota, we will not discuss it here.

Chips

After working with these two previous continuous models, chips can be introduced as a discrete model. While using the chip model, teachers should use chips that are two-colored, with a light color on one side and a dark color on the other (Behr & Post, 1992). The chip model could be used to show a fractional part of a set of objects. If a student is asked to show 2/3 of a set of 12 chips, the student would first need to take the 12 chips and sort them into three equal groups. The student would then take two of these groups and flip them to the darker color to represent the fractional amount desired (eight chips). Another way to model 2/3 of a set of 12 chips would be to sort chips into groups of three chips. The student would then flip two chips in each group to the darker color and count all darker colored chips.

To add fractions with chips, students would have to find a common number of chips (the unit) to model both fractions (Cramer & Wyberg, 2009). Then, students would turn over the number of chips used to show each fraction, and combine the turned over chips. After, students would compare the number of turned over chips to the number of chips used to show the unit.

Comparing fractions

In an 18 week investigation performed by Post, Behr, and Lesh (1986), fourth grade students were asked to compare fractions to show understanding of order and equivalence of rational numbers. Based on this study Behr and Post (1992) suggest a process for teaching comparing fractions. First students should work with manipulatives to compare unit fractions (fractions that have a numerator of one). After extensive practice with physical materials, students should be able to conceptualize the relationship between the sizes of parts compared to the number of parts it takes to make a whole (Behr & Post, 1992).

After a relationship has been identified, students should move on to comparing fractions with the same denominator, such as 2/5 and 4/5. Students who were able to successfully compare these types of fractions recognized the size of the pieces were the same in both fractions. However, one of the fractions had more pieces and, thus, was the greater fraction (Behr & Post, 1992).

From here, researchers recommend comparing fractions with the same numerator, such as 5/7 and 5/9. Students who were successful comparing these types of fractions were able to identify that both fractions have the same number of pieces while understanding that ninths were smaller than sevenths. Since they both had the same number of pieces and ninths were smaller pieces, 5/9 must be less than 5/7 (Behr & Post, 1992).

The researchers identified two strategies commonly being reported without being specifically taught when comparing fractions that had neither the same numerator nor denominator: residual thinking and benchmarking. Residual thinking refers to students looking at how much more is needed to complete the unit (Post & Cramer, 2002). For example, in comparing the fractions 5/6 and 7/8, students are able to declare that 7/8 is greater because the size of the piece needed to make a unit (1/8), is smaller than it would need to be with 5/6 (1/6). Since a person would need to add a smaller piece to make a unit, it is closer to making a whole unit and must be greater.

Benchmarking refers to a situation where the transitive property is used compare a given fraction to an external value, the benchmark fraction (Post, Behr, & Lesh, 1986). Common reference points which have been successfully used by students to compare fractions have been 0, ½, and 1 (Clark & Roche, 2009; Petit et al., 2010). For example, when comparing the fractions 4/9 and 6/11, students are able to say 4/9 is less than ½ because 4/8 is equivalent to ½ and ninths are smaller than eighths. Six-elevenths is greater than ½ because 6/12 is equivalent to ½ and elevenths are bigger than twelfths, so 6/11 must be greater than 4/9.

Chapter 3: Interpretation

The research reviewed in this paper covers many topics within the realm of rational numbers. There are at least five conceptions of fractions and students need to experience each type of fraction concept in order to become successful. Research has reported common misconceptions when it comes to fractions that can be categorized into two main areas: rote learning and complexity of the concept. There needs to be a time commitment to allow students to conceptualize what fractions are and, only after this has been done, can students move on to operations with fractions. Research has found one of the difficulties students experience in understanding rational numbers is that fractions do not follow the same patterns they have been accustomed to working with throughout their educational career. It is important for teachers to scaffold students with real-world problems that allow students to use problem-solving skills. There are many different fraction manipulatives to use with students, but four of the most common are fraction circles, paper folding, number lines, and chips. Studies have shown that providing students meaningful opportunities to practice with these manipulatives will aid students in comparing fractions, as well as other operations that students are expected to have learned by the time they leave fifth grade.

As a teacher, teaching fractions in this way has been a new process for the author. I have taught the concepts of fractions to my students using the Rational Number Project materials exclusively for three years. What I have seen from my students matches what I have found in the research. In this section I will:

- describe my classroom and school district,

- discuss the background of the RNP curriculum and the benefits of teaching my students with it,
- discuss the mistakes I have made teaching fractions, and
- describe my general process for teaching fraction concepts and skills.

My Classroom

I teach in a school district on a Native American reservation in northern Minnesota. Our school district is comprised of five buildings and has a total population of 1,398 students. My building has a population of 498 students in grades first through fifth, 100% of which are Native American and 89% qualify for free and reduced lunch. I teach one of the five sections of fifth grade in the building. I have had between fourteen and twenty-four students in my classroom, varying in abilities from second grade to ninth grade mathematics level based on Measures of Academic Progress scores.

RNP Background and My Benefits of Using These Lessons

The Rational Number Project is a cooperative research project that has been funded by the National Science Foundation. Researchers Merlyn Behr, Kathleen Cramer, Thomas Post, and Richard Lesh spent many years researching how students learn fractions successfully by observing lessons and interviewing students. The RNP philosophy is that extended periods of time devoted with manipulative materials developing concepts, order, and equivalence are needed before students can operate on fractions in a meaningful way.

The RNP Level 1 lessons were piloted in the Rosemount-Apple Valley, MN school district (Cramer, Behr, Post & Lesh, 1997). Thirty-three fourth- and fifth-grade teachers used the RNP lessons while thirty-three other fourth- and fifth-grade teachers used the district's adopted textbook series (Cramer et. al, 1997). RNP students outperformed textbook students in all areas assessed. Specific differences were found in students' thinking. RNP students thought about fractions in a conceptual manner, while textbook students had a habit of thinking about fractions procedurally. RNP students were better able to verbalize their thinking about fractions than the textbook students.

Out of the 23 lessons in the first RNP unit, only five deal specifically with fraction addition and subtraction. Despite the limited time spent on fraction operations, assessments in this piloted study showed that RNP students performed as well as students using traditional texts on fraction addition and subtraction problems, while consistently outperforming other students on estimation assignments involving fraction addition and subtraction (Cramer, Post, & delMas, 2002).

Children learning with RNP lessons use a variety of manipulatives. They work in small groups discussing ideas and collaborate with their teacher in large group settings. They draw pictures to record different models. They solve story problems using manipulatives to model actions in stories. This model for teaching reflects the theoretical framework suggested by Jean Piaget, Jerome Bruner, and Zoltan Dienes.

In these lessons, students solve story problems for fraction operations using pictures. They record their actions with pictures as number sentences. This is an example of a real world to picture to symbol translation, as seen in figure 1.

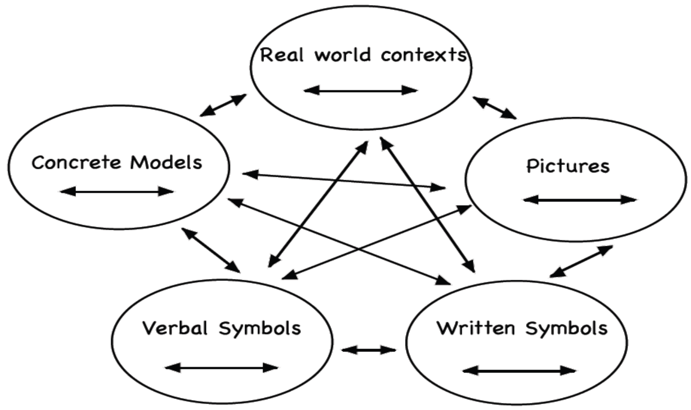

Figure 1. Richard Lesh's Instructional model which focuses on making connections between different representations (Kramer, et. al 1997)

After teaching RNP lessons for three years, I can see many benefits with my students, including higher confidence in fractional abilities. I have noticed increased MAP scores in rational number skills, and higher rates of proficiency on Minnesota MCA assessments. Lamon has stated teachers need a multitude of learning tools to reach all students successfully. Students in my classroom are introduced to a variety of manipulatives and given a chance to explore how each manipulative can be used to represent rational numbers. After experiences with each model, students compare and contrast the featured model to models previously learned. By the end of the fractions unit, students have a multitude of models to fall back on if they reach a problem that is unfamiliar to them.

Lesh, Post, & Behr (1987) have stated students should be striving for transferability in fractions knowledge between fractions constructs. My students have been able to communicate about their solution strategies. Students decipher which manipulative would work best for them and understand how to use the manipulative to successfully solve the problem.

Research from Lamon (2001) has shown that students who think about fractions conceptually outperform students who think about this same concept procedurally. After working extensively with fraction circles and discussing relationships between the different colored pieces, students are able to give multiple names for each fraction because they can see what each fraction looks like. They understand that there is more than one way to show a partial amount and, by cutting into smaller pieces, we change the name of the fraction without changing the shaded amount. For example, three red pieces (sixths) of the circle cover the same amount as two blue pieces (fourths). This has been a huge help when we reach the concept of adding and subtracting fractions. Students now know we need to make our pieces the same size before we can do anything and are able to determine how to get each fraction into the same-sized pieces. They understand what this process looks like and it makes sense.

Mistakes Made While Teaching with RNP

As with many others, I have made mistakes while teaching fraction concepts. Research from Charalambos, Charalambous, and Demetra Pitta-Pantazi (2007) states that a disproportionate amount of time is spent with the part-whole construct of fractions. I am one of the teachers who has spent too much time focusing on this realm. Prior to researching this topic, I did not know the different types of fractions. I thought fractions were fractions and that if you knew how to perform operations with a type of fraction, you were successful. My assessments showed me this is not the case. Students were unable to successfully answer questions relating to measurement, which was predominantly seen on standardized testing.

I have also had a difficult time dedicating the necessary time to successfully teach fractions. Lamon (2001) has stated that, even after 18 weeks of instruction, students can still be uncomfortable with fraction operations. I start teaching fractions in January and, due to pressure from the MCA's, had usually allotted approximately three weeks to cover the topics required by the Minnesota Academic Standards for Mathematics (M.N. Department of Education, 2010). However, after those three weeks had come and gone, I found myself going back to teach remedial skills.

I also rushed through conceptualizing what a fraction is. I found several students showing hesitance to use the manipulatives in front of them because they said it was boring. I took this as them already having mastered this topic when it was not true. These same students would come to the skill of finding equivalents and would try to use an algorithm that was incorrect only to become dejected.

A General Process for Teaching Fractions Skills

As I introduce fractions to my students, I first want to see what background knowledge they have coming in. I start by giving my students the written test for RNP 1. I then spend time correcting these assessments so I know which areas of instruction to emphasize.

As has already been discussed, research performed by Lamon (2001) states that students who are taught with conceptualization outperform students who are taught procedurally. The class practices conception showing multiple fractions with fraction circles. We compare fractions using these manipulatives and discuss how we know which fractions are greater.

Studies performed by Lamon (2001) also show that multiple representations will reach different learning styles and more students. After we practice showing fractions with fraction circles, we introduce paper folding as another strategy for showing rational numbers. We create paper strips which are the same size and practice making folds to partition our strips. We practice making each denominator and discuss when this strategy would be successful. Then, we compare and order fractions using paper strips.

Next, number lines are introduced. By this time, students have become somewhat familiar with the concept of part-whole fractions and are ready to take a step toward abstraction. We discuss how to show fractions on a number line and compare number lines to our other manipulatives thus far. I give the students real-world application situations where number lines are commonly used. We then explore how precise number lines allow us to get and talk about this manipulative's strengths. We show multiple fractions and compare them using separate number lines.

Then, we practice showing fractions of a set. The chip model is introduced and students practice partitioning groups of objects. We then discuss that, instead of counting pieces (fraction circles), sections (paper folding), or jumps (number line), we now count the number of groups desired. After figuring out how many groups are desired, we then count the total number of objects in those groups. Students become a little flustered at this stage because of the unfamiliarity they have with this type of fraction. I can really relate to the research from Petit, Laird, & Marsden (2010) indicating some teachers solely focus on construct of fractions and that a disproportionate amount of time is spent on the part-whole idea of fractions. We spend extra time with this model and parallel our thinking to previous experiences of modeling division word problems. After students have become comfortable showing fractions of a set we start comparing fractions using different sets.

Up to this point, students have been introduced to four different manipulatives, but have focused on the same skills with each manipulative. Research states that often times teachers rush into performing abstract operations before students are cognitively ready (Charron, 2002). This is why we focus solely on looking at what each fraction is and only compare them to one another. It is my hope that, by this juncture, students have become comfortable with at least one manipulative, and have been comparing and contrasting models to try to understand how each works, building transferability. Students have focused on the manipulatives in front of them and have had little practice writing symbols.

After several weeks of practice, having students develop successful strategies for comparing fractions, we then start to combine our fractions. We discuss estimates of our combined fractions to practice reasoning. Afterward, we discuss how we can determine what the exact answer would be. We then discuss the importance of a common denominator (comparing both fractions to a colored piece that will cover both fractions). We then spend class periods practicing finding common denominators using a model in which each student is comfortable, then having students present to the class how they were able to successfully perform the task. We show each model for all problems so students who are having difficulty with a model will be able to link it with one they have already mastered. The goal is to build transferability throughout this process.

The last concept I introduce is subtraction of fractions. We approach this skill through real-world experiences with which students are familiar. We constantly reach back to the conceptualization piece with what each fraction looks like. We estimate what our answers will look like before performing any operations. Students then choose a model with which they are comfortable and show the operation using a manipulative. Presentations are given to classmates and multiple solution paths are encouraged.

I have noticed that by teaching fractions in this manner, constantly stressing conceptualization and using manipulatives, I do not need to teach finding least common multiples for finding common denominators or greatest common factors for simplifying fractions. Students have constructed mental images through experiences that are retained throughout the school year. Meaning is put to the process and I am asked "Why do we do that?" less frequently.

Chapter 4: Conclusion

The amount of research performed on fractions is abundant. However, more still needs to be done. Below each research question will be answered with the research reviewed. Then the paper will discuss what still needs to be researched and plans for the use of the research.

Types of Fractions

1. What does each type of fraction entail?

Fractions can currently be categorized into at least five interrelated categories. These categories are labeled: part-whole, ratio, operator, quotient, and measure. Each of these categories requires students to look at a fraction in a different manner. Part-whole fractions require students to look at a fraction as one object partitioned into a number of pieces. The ratio aspect requires students to look at a fraction as a comparative relationship between the numerator and the denominator. The operator facet asks students to relate inputs (numerators) to outputs (denominators). The quotient construct requires students to view a fraction as a division problem, where the numerator is divided by the denominator. The measure category requires students to view fractions as a distance from one point to another.

2. What foundations and skills do students need in order to show mastery of each concept?

In order to master the part-whole focus of fractions, students need to understand that when comparing or performing operations, both units must be the same size and the unit needs to be broken into equal-sized parts. Students also need to be able to perform the partitioning. Students should understand the sum of the parts is equal to the unit.

To master the ratio aspect of fractions, students need to build the idea of relative amounts. Students should know that when the two amounts being compared (the numerator and denominator) change, they change together, leaving the relationship the same. This is where fraction equivalence comes from and equivalence must be developed before students can fully grasp this realm.

For the operator category, students should be able to look at a fraction as a fractional multiplier (i.e. 2/3 could be looked at as 2 x 1/3). Students should also be able to relate inputs and outputs (i.e. 4/3 could be looked at as having put in 4 objects and watching them come out in 3 groups).

For the quotient construct, students need to be able to associate fractions with division and recognize the role of the dividend and the divisor in the operation. Mastering this construct requires students to develop an understanding of partitive and quotitive division.

For the measure concept, students need to be able to look at a fraction and identify its unit fraction (1/a). The student should then be able to repeat the unit fraction as many times as needed to get from the starting point (0) to the desired fractional amount. Students develop an understanding of the rational number property of density.

Common Misconceptions Encountered During Instruction

There have been many mistakes reported in research, ranging from teaching styles to time spent on the topic. There have been a couple of misconceptions that have been repeatedly cited in research, however. I will now go through some of the most common errors made by students and teachers while learning/teaching about rational numbers.

A common error made by students learning fractions, reported by Behr & Post (1988), is they see a fraction as two whole numbers instead of a single quantity. If looking at fractions in this regard, comparing fractions becomes very difficult, as well as adding and subtracting fractions. Students who show this type of error have not conceptualized fractions and need experiences using concrete manipulatives.

From a teaching standpoint, a common error reported by Behr, Wachsmuth, Post, & Lesh (1984) and Saenz-Ludlow (1985) is that teachers often teach about fractions abstractly and procedurally, while students are not prepared for that type of instruction. NCTM (1989) has advised that conceptual understanding be developed before computational fluency. Lamon (2001) has stated students who have been taught conceptually outperformed students taught procedurally in all aspects of fraction learning. One reason for this finding is because of students' use of incorrect whole-number counting strategies. When fractions instruction comes into play, reverting back to other strategies will not work. This sets students up for failure.

Another common error in fraction instruction is introducing students to fraction operations before students are ready. Students need time to conceptualize fractions, but educators typically spend more than 85% of instruction time focusing on the operations of fractions. Huinker (1998) has stated that when students are not confident in an algorithm, they revert back to strategies that make sense to them, which are not consistent from fractions to whole numbers.

Also, teachers often focus on a limited number of constructs while teaching fractions. Petit, Laird, & Marsden (2010) reported that teachers spend a disproportionate amount of time focusing on the part-whole aspect of rational numbers, even though Lamon (2001) reported this realm is not sufficient as a foundation. Petit, Laird, & Marsden (2010) have also suggested that this limited-opportunity style of instruction may reinforce inappropriate whole-number reasoning.

Instructional Best Practices for Grades 3, 4, and 5 Fraction Concepts

Instructional best practices call for teachers to reorganize their teaching strategies from many currently-used approaches. Aksu (1997) has stated students need to work with fractions in a hands-on manner, focusing on real-world situations for a great deal of time before any emphasis should be placed on abstraction. I will now go through some specific recommendations research has provided.

Lamon (2001) stated students who have conceptualized fractions have higher success rates on assessments. Students who conceptualized rational numbers no longer need to refer back to whole-number strategies. They can look at a fraction as a single quantity. While building students' conceptualization, teachers should focus on introducing a variety of manipulatives. A variety will not only reach multiple types of learners, but will also introduce students to numerous constructions of fractions. Conceptualization should take precedence over teaching procedures and operations.

The NCTM (1989) stated the goal of mathematics instruction should be learning to solve problems. Because of this, students should be frequently exposed to problem-solving situations. These situations create meaning for students and set a purpose for performing operations. Problem-solving situations have been linked to the self-discovery of many learning strategies, including traditional algorithms.

Successful instruction takes a great deal of time. In Minnesota, roughly 40% of questions on standardized tests relate to fraction concepts (M.N. Department of Education, 2010). Instruction time should mirror this importance. Research performed by Lamon (2001) demonstrated that confidence in this concept takes up to two years. We are reconfiguring students' thought processes and should plan instruction accordingly.

A Call for More Research

An important part of effectively teaching fractions is proper professional development for educators. At what point in a teacher's career are instructional habits formed? For me, during the first three years I was continuously exploring my environmental surroundings as well instructional practices. I believe many educators endlessly search for ways to improve their instruction for the greater good of their students. However, with a large percentage of teachers being taught fractions procedurally and abstractly, many educators have insufficient knowledge and resources to effectively teach the material for which they are responsible.

There is a large amount of research on fractions. However, I did not find much on teaching fractions to different ethnicities. Do different parts of the world learn differently? Do certain manipulatives have a higher efficiency for some cultures compared to others? I am interested in finding out if there is a certain methodology of teaching mathematical lessons to my students that is more successful than methods I currently utilize.

Future Use

This research will be tremendously important in my teaching career. I have found models and manipulatives that help students learn different constructs of fractions. I have discovered types of fractions I did not know about. Upon discussing my paper with colleagues, I have received several questions regarding fractions instruction and this topic is now a focal point of mathematics instruction for grades three through five in our school. Our building has decided to teach fractions using RNP lessons exclusively. I have given presentations about RNP philosophy as well as common errors in student thinking while comparing fractions. I have shown models that have been proven to be successful with students and teachers have been receptive to this information. As grade-level teams, we have set up a scope and sequence using RNP: Initial Fraction Ideas and RNP: Fraction Operations and Initial Decimal Ideas. We have become more cohesive as a school regarding this instruction and have developed a plan for making this a successful topic for our students.

Upon completion of this paper, I will share new findings with my grade-level team and others in grades three through five at the school where I teach. I will present this information to our Math Facilitator and will be able to send resources to her that can be distributed to all teachers. With a lack of time to delve during the school year, I can provide information to teachers who are looking for ways to improve their fraction instruction so their students have more opportunities to be successful. I plan to explore traveling to state and local conferences to share the information I have gathered with other educators interested in improving their mathematics instruction within the concepts of fractions.

Bibliography

Aksu, M. (1997). Student performance in dealing with fractions. *Journal of Educational Research*, *90*, 375-380.

Baturo, A.R. (2004). 'Empowering Andrea to help year 5 students construct fraction understanding', *Proceedings of the 28th PME Conference*, Vol. 2, Bergen University College, Bergen, pp. 95-102.

Behr, M., Harel, G., Post, T. and Lesh, R, (1993). 'Rational number: Toward a semantic analysis-emphasis on the operator construct', in T.P. Carpenter, E. Fennema and T.A. Romberg (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, New Jersey, pp. 13-47.

Behr, M., & Post, T. (1988). Teaching Rational Number and Decimal Concepts. In T. Post, (Ed.), *Teaching Mathematics in Grades K-8: Research Based Methods* (pp. 190-231). Newton, MA: Allyn & Bacon, Inc.

Behr, M., & Post, T., (1992). Teaching rational number and decimal concepts. In T. Post (Ed.), *Teaching mathematics in grades K-8: Research-based methods* (2nd ed.) (pp. 201-48). Boston: Allyn and Bacon.

Behr, M. J., Leash, R., Post, T. R., & Silver, E. A., (1983). Rational-number concepts. In R. Lesh ., & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 92-126). New York: Academic Press.

Behr, M., Wachsmuth, I., Post T., & Lesh R. (1984, November). Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment. Journal for Research in Mathematics Education, 15(5), 323-341.

Bezuk, N., & Cramer, K. (1989). Teaching about fractions: What, when, and how? In P. Trafton (Ed.), *New directions for elementary school mathematics: 1989 yearbook* (pp. 156-167). Reston, VA: National Council of Teachers of Mathematics.

Bright, G., Behr, M., Post, T., & Wachsmuth, I. (1988). Identifying fractions on number lines. *Journal for Research in Mathematics Education,* 19(3), 215-232.

Carpenter, T. P.. Kepner, H., Corbitt, M. K., Lindquist, M. M., & Reys, R. E. (1980). Results and implication s of the Second NAEP Mathematics Assessments: Elementary school. *Arithmetic Teacher,* 2(8), 10-13.

Carraher, D.W. (1996). 'Learning about fractions', in L.P. Steffe, P. Nesher, P.Cobb, G.A. Goldin and B. Greer (eds.), *Theories of Mathematical Learning*, Lawrence Erlbaum Associates, New Jersey, pp. 241-266.

Chan, W., Leu, Y., & Chen, C. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. *Journal of Experimental Education, 76*(1), 26-57.

Charalambous, C.Y., & Pitta-Pantazi, D. (2005) Revisiting g a theoretical model on fractions: Implications for teaching and research. In Chick, H.L. and Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group ofor the Psychology of Mathematics Education,* Vol. 2, pp. 233-240. Melbourne: PME.

Charalambous, C., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics, 64*(3), 293-316.

Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educ Stud Math*, *72*(1), 128-137.

Charron, C. (2002). Conceptualization of fractions and categorization of problems for adolescents. *European Journal of Psychology of Education, 17*(2), 115-128.

Cramer, K., & Henry, A. (2002). Using manipulative models to build number sense for addition of fractions. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions: NCTM 2002 yearbook*, (pp. 41 – 48). Reston, Va: NCTM.

Cramer, K., Behr, M., Post, T., & Lesh, R. (1997). Teacher's guide. In *Rational Number Project: Fraction Lessons for the Middle Grades Level 1* Dubuque, IA: Kendall Hunt Publishing

Cramer,K., Post, T., & del Mas, R. (2002). Initial fraction learning by fourth- and fifth- grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education, 33* (2), 111-144.

Cramer, K., & Wyberg, T. (2009). Efficacy of different concrete models for teaching the part-whole construct for fractions. *Mathematical Thinking and Learning: An International Journal, 11*(4), 226-257.

Dienes, Z. (1960). *Building up mathematics*. London, England: Hutchinson Educational LTD.

Huinker, D. M. (1998). Letting fraction algorithms emerge through problem solving. *National Council of Teachers of Mathematics Yearbook* , *1998*, 170-182.

Keijzer, R., & Terwel, J. (2004). A low -achiever's learning process in mathematics: shirle'ys fraction learning. *Journal of Classroom Interaction*, *39*(2), 10-23.

Kieren, T.E. (1993). ‘Rational and fractional number: From quotient fields to recursive understanding’, in T.P. Carpenter, E. Fennema and T.A.Romberg (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, NJ, pp. 49-84.

Lamon, S. (1993). ‘Ratio and proportion: Children’s cognitive and metacognitive process’, in T.P.Carpenter, E. Fennema and T.A. Romberg (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, New Jersey, pp. 131-156.

Lamon, S. (1999). *Teaching Fractions and Ratios for Understanding*, Lawrence Erlbaum Associates, New Jersey.

Lamon, S. (2001). Presenting and representing: From fractions to rational numbers. *Yearbook- National Council of Teachers of Mathematics*, *2001*, 146-165.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*, 33-40. Hillsdale, New Jersey & London: Lawrence Erlbaum.

Marshall, S.P. (1993). ‘Assessment of rational number understanding: A schema-based approach’, in T.P. Carpenter, E. Fennema and T.A. Romberg (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, New Jersey, pp. 261-288.

Martin, M. O., Mullis, I. V., & Foy, P. (2008). *TIMSS 2007 international mathematics report: Findings from IEA's trends in international mathematics and science study at the fourth and eighth grades*. Boston, MA: TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College.

M.N. Department of Education, Division of Research and Assessment. (2010). *Mathematics test specifications for mca-iii, grades 3-8 and modified, grades 5-8.*

Moss, J. & Case, R. (1999). Developing Children’s Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journalfor Research in Mathematics Education* 30(2). 122-147.

National Council of Teachers of Mathematics (NCTM). *A History of Mathematics Education in the United States and Canada.* Thirty-second Yearbook of the NCTM. Washington, D.C.: NCTM, 1970.

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, Va.: National Council of Teachers of Mathematics, 1989.

Niemi, D. (1995). *Instructional influences on content area explanations and representational knowledge: Evidence for the construct validity of measures of principled understanding mathematics* (CSE Technical Rep. No. 403). Los Angeles: University of California, National Center for Research on Evaluation, Standards, and Student Testing.

Niemi, D. (1996). A fraction is not a piece of pie: Assessing exceptional performance and deep understanding in elementary school mathematics. *Gifted Child Quarterly*, *40*(2), 70-80.

Orton, R., Post, T., Behr, M., Harel, G., & Lesh, R. (1995). Logical and psychological aspects of rational number pedagogical reasoning. *Hiroshima Journal of Mathematics Education*, 3, 63-75.

Petit, M. M., Laird, R. E., & Marsden, E. L. (2010). *A focus on fractions: Bringing research to the classroom*. (pp. 33-35). New York, NY: Routledge.

Post, T. (1992). Some notes on the nature of mathematics learning. In T. Post (Ed.), *Teaching mathematics in graders K-8; Research-based methods* (pp. 1-22). Needham, MA: Allyn and Bacon.

Post, T., Behr, M. J., & Lesh, R. (1986). Research-based observations about children's learning of rational number concepts. *Focus on Learning Problems in Mathematics, 8*(1), 39-48.

Post, T., & Reys, R. E. (1979). Abstraction Generalization and Design of Mathematical Experiences for Children. In K. Fuson & W. Geeslin (Eds.), *Models for mathematics learning.* (pp. 117-139). Columbus, OH: ERIC/SMEAC.

Roche, A., & Clarke, D. (2009). Making sense of partitice and quotitive division: A snapshot of teachers' pedagogical content knowledge. *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australia*, *2*,

Romberg, T. A., & Carpenter, T. P. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In W. C. Merlin (Ed.), (3rd ed.,pp. 850-73). *Handbook of Research on Teaching*. London: Collier Macmillan.

Saenz-Ludlow, A. (1995, March). Ann's fraction schemes. *Educational Studies in Mathematics, 28*(2), 101-132.

Saxe, G. B., Shaughnessey, M., Shannon, A., Langer-Osuna, J., Chinn, R., &Gearhart, M. (2007). Learning about fractions as points on a number line. In W.G. Martin, M.E. Srutchens, & P. C. Eliot (Eds.), *The learningof mathematics 2007 yearbook* (pp. 221-236). Reston, VA: National Council of Teachers of Mathematics.

Streefland, L. (1993). ‘Fractions: A realistic approach’, in T.P.Carpenter, E. Fennema, and T.A. Romberg (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, New Jersey, pp. 289-325.